Math 113 (Calculus II)
Test 3 Form A KEY

Multiple Choice. Fill in the answer to each problem on your scantron. Make sure your name, section and instructor is on your scantron.

1. Which definite integral represents the length of the arc given by $y = 2 + 3\sin x$ for $0 \le x \le 4$?

a) $\displaystyle\int_0^4 \sqrt{1 + 3\cos^2 x}\, dx$ b) $\displaystyle\int_0^4 \sqrt{1 + 9\cos^2 x}\, dx$

c) $\displaystyle\int_0^4 \sqrt{1 + 3\sin^2 x}\, dx$ d) $\displaystyle\int_0^4 \sqrt{1 + 9\sin^2 x}\, dx$

e) $\displaystyle\int_0^4 \sqrt{1 + (2 + 3\cos x)^2}\, dx$ f) $\displaystyle\int_0^4 \sqrt{1 + (2 + 3\sin x)^2}\, dx$

g) $\displaystyle\int_0^4 \sqrt{1 + (2x - 3\cos x)^2}\, dx$ h) none of the above

Solution: b)

2. What is the area of the surface generated when the arc given by $y = \sqrt{1 - x^2}$ for $0 \le x \le 1$ is revolved around $y = 1$?

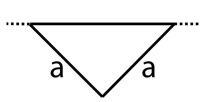
a) $\frac{1}{2}\pi$ b) π c) 2π

d) $\pi(\pi + 1)$ e) $\pi(\pi - 2)$ f) $\frac{1}{2}\pi^2$

g) π^2 h) none of the above.

Solution: e)

3. A plate in the shape of an isosceles triangle of side lengths a, a, and $\sqrt{2}a$ meters is suspended in a liquid so that the long side of the triangle is horizontal and just submerged (see picture). If the density of the liquid is δ kilograms per cubic meter, what is the force due to fluid pressure on one side of the plate?



a) $\dfrac{9.8\delta\sqrt{2}a^3}{24}$ b) $\dfrac{9.8\delta\sqrt{2}a^3}{12}$ c) $\dfrac{9.8\delta\sqrt{2}a^3}{6}$

d) $\dfrac{9.8\delta\sqrt{2}a^3}{3}$ e) $\dfrac{9.8\delta\sqrt{2}a^3}{2}$ f) $9.8\delta\sqrt{2}a^3$

g) None of the above.

Solution: b)

4. The area inside a circle of radius 2 centered at $(1, 3)$ is rotated about $x = -4$. The volume of the resulting solid is

a) 2π b) $5\pi^2$ c) $10\pi^2$

d) $20\pi^2$ e) $40\pi^2$ f) $60\pi^2$

g) None of the above.

Solution: e)

5. We say $\lim_{n\to\infty} a_n = L$

a) if for every $\epsilon > 0$, there is an integer N such that if $n > N$, then $|a_n - L| < \epsilon$.

b) if for some $\epsilon > 0$, there is an integer N such that if $n > N$, then $|a_n - L| < \epsilon$.

c) if for some integer N, there is an $\epsilon > 0$ such that $|a_n - L| < \epsilon$ whenever $n > N$.

d) if for some integer N, $|a_n - L| < \epsilon$ for every $\epsilon > 0$.

e) if for every integer N, there is an $\epsilon > 0$ so that if $n > N$, then $|a_n - L| < \epsilon$.

f) None of the above.

Solution: a)

6. A sequence $\{a_n\}$ is defined by $a_1 = 1$ and $a_{n+1} = 4 - \dfrac{1}{a_n}$ for $n \geq 1$. Assuming that the sequence is convergent, find its limit.

a) 4 b) 3 c) $2 + \sqrt{3}$

d) $2 - \sqrt{3}$ e) $4 + 2\sqrt{3}$ f) None of the above.

Solution: c)

7. $\lim_{n\to\infty} \dfrac{n!}{n^n} =$

a) 0 b) 1 c) -1

d) e e) diverges f) None of the above

Solution: a)

8. $\displaystyle\sum_{n=2}^{\infty} \frac{2}{n^2-1} =$

a) 0 b) 1 c) $\frac{3}{2}$

d) $\frac{1}{2}$ e) diverges f) None of the above

Solution: c)

Free response: Give your answer in the space provided. Answers not placed in this space will be ignored.

9. (8 points) Find the length of the arc given by $y = \ln(\cos x)$ for $0 \le x \le \frac{\pi}{4}$.

Solution: Since

$$y' = \frac{1}{\cos x} \cdot (-\sin x) = -\tan x,$$

we see that the length is given by

$$\int_0^{\pi/4} \sqrt{1+\tan^2 x}\, dx = \int_0^{\pi/4} \sec x\, dx$$

$$= \ln|\sec x + \tan x|_0^{\pi/4} = \ln(\sqrt{2}+1)/.$$

10. (8 points) Find the area of the surface generated when the arc given by $y = 2+x^2$ for $1 \le x \le 3$ is revolved around the y-axis.

Solution: Since the radius of revolution about the y axis is x, and $y' = 2x$, we have

$$\int_1^3 2\pi x\sqrt{1+4x^2}\, dx.$$

Letting $u = 1+4x^2$, $du = 8x\, dx$, or $x\, dx = du/8$. The above integral becomes

$$\int_5^{37} 2\pi u^{1/2}\frac{du}{8} = \frac{\pi}{4}\cdot\frac{2}{3}u^{3/2}|_5^{37}$$

$$= \frac{\pi}{6}((37)^{3/2} - 5^{3/2}).$$

11. (8 points) Find the centroid of the finite region between $y = \sqrt{x}$ and $y = x^3$.

Solution: Notice that the finite region between the functions satisfies $0 \le x \le 1$.

$$A = \int_0^1 (\sqrt{x} - x^3)\, dx = (\frac{2}{3}x^{3/2} - \frac{x^4}{4})|_0^1 = \frac{2}{3} - \frac{1}{4} = \frac{5}{12}.$$

$$M_y = \int_0^1 x(\sqrt{x} - x^3)\,dx = \int_0^1 (x^{3/2} - x^4)\,dx = (\frac{2}{5}x^{5/2} - \frac{x^5}{5})|_0^1 = \frac{1}{5}.$$

$$M_x = \int_0^1 \frac{1}{2}(x - x^6)\,dx = (\frac{x^2}{4} - \frac{x^7}{14})|_0^1 = \frac{5}{28}.$$

$$\overline{x} = \frac{\frac{1}{5}}{\frac{5}{12}} = \frac{12}{25}, \quad \overline{y} = \frac{\frac{5}{28}}{\frac{5}{12}} = \frac{12}{28} = \frac{3}{7}.$$

12. The nth partial sum of a series $\sum_{i=1}^{\infty} a_i$ is $s_n = \dfrac{n-2}{n+2}$.

(a) (2 points) Find $\sum_{n=1}^{\infty} a_n$.

Solution:

$$\sum_{n=1}^{\infty} a_n = \lim_{k\to\infty} s_k = \lim_{k\to\infty} \frac{k-2}{k+2} = 1.$$

(b) (1 points) Find a_1.

Solution:

$$a_1 = s_1 = \frac{1-2}{1+2} = -\frac{1}{3}.$$

(c) (3 points) Find a_n for $n > 1$.

Solution:

$$a_n = s_n - s_{n-1} = \frac{n-2}{n+2} - \frac{n-3}{n+1} = \frac{(n-2)(n+1) - (n-3)(n+2)}{(n+2)(n+1)}$$

$$= \frac{n^2 - n - 2 - (n^2 - n - 6)}{(n+2)(n+1)} = \frac{4}{(n+2)(n+1)}.$$

13. (8 points) Find the values of x for which the series $\sum_{n=1}^{\infty} \dfrac{x^{2n}}{2^{2n}}$ converges. Find the sum of the series for those values of x.

Solution:

$$\sum_{n=1}^{\infty} \frac{x^{2n}}{2^{2n}} = \sum_{n=1}^{\infty} \left(\frac{x^2}{4}\right)^n$$

is a geometric series, but is missing the first term. It converges when the ratio is less than 1, or when

$$\frac{x^2}{4} < 1.$$

This leads to $|x| < 2$.

$$\sum_{n=1}^{\infty}\left(\frac{x^2}{4}\right)^n = \sum_{n=0}^{\infty}\left(\frac{x^2}{4}\right)^n - 1$$
$$= \frac{1}{1-\frac{x^2}{4}} - 1 = \frac{1-(1-\frac{x^2}{4})}{1-\frac{x^2}{4}} = \frac{x^2}{4+x^2}.$$

14. (8 points) Use the integral test to determine if $\displaystyle\sum_{n=1}^{\infty} ne^{-n}$ converges.

Solution: We use integration by parts to integrate $\displaystyle\int xe^{-x}\,dx$. Let $u = x$. Then $du = dx$, and $dv = e^{-x}\,dx$. Thus, $v = -e^{-x}$.

$$\int xe^{-x}\,dx = -xe^{-x} + \int e^{-x}\,dx = -xe^{-x} - e^{-x} + C.$$

Hence,

$$\int_1^{\infty} xe^{-x}\,dx = \lim_{b\to\infty}\int_1^b xe^{-x}\,dx = \lim_{b\to\infty}(-xe^{-x} - e^{-x})|_1^b$$
$$= \lim_{b\to\infty}(-\frac{b}{e^b} - \frac{1}{e^b} + \frac{1}{e} + \frac{1}{e}) = \frac{2}{e}.$$

Since the improper integral converges, the series also converges by the integral test.

15. (6 points) Find $\displaystyle\lim_{n\to\infty}\{\ln(n+1) - \ln n\}$.

Solution:

$$\lim_{n\to\infty}\{\ln(n+1) - \ln n\} = \lim_{n\to\infty}\ln\left(\frac{n+1}{n}\right) = \ln\left(\lim_{n\to\infty}\frac{n+1}{n}\right) = \ln(1) = 0.$$

16. (8 points) How many terms in the series $\displaystyle\sum_{n=2}^{\infty}\frac{1}{n(\ln n)^2}$ would you need to add to find the sum to within $\dfrac{1}{\ln 100}$?

Solution: The error of truncating after the nth term is bounded by

$$\int_n^{\infty}\frac{1}{x\ln^2(x)}\,dx.$$

If we let $u = \ln x$, then $du = \dfrac{1}{x}\,dx$, and

$$\int_n^{\infty}\frac{1}{x\ln^2(x)}\,dx = \lim_{b\to\infty}\int_n^b\frac{1}{x\ln^2(x)}\,dx = \lim_{b\to\infty}\int_{\ln n}^{\ln b}\frac{1}{u^2}\,du$$
$$= \lim_{b\to\infty}(-\frac{1}{u})|_{\ln n}^{\ln b} = \lim_{b\to\infty}(-\frac{1}{\ln b} + \frac{1}{\ln n}) = \frac{1}{\ln n}.$$

Thus, we would want

$$\frac{1}{\ln n} \leq \frac{1}{\ln 100}$$

and $n \geq 100$. Note that this means we would sum 99 terms, because the summation starts at 2.

17. (8 points) Determine if $\sum_{n=1}^{\infty} \frac{n-1}{n^2\sqrt{n}}$ converges or diverges. Explain your reasoning.

Solution: For large n,

$$\frac{n-1}{n^2\sqrt{n}}$$

behaves like $\frac{1}{n^{3/2}}$. We can use the limit comparison test.

$$\lim_{n\to\infty} \frac{\frac{n-1}{n^2\sqrt{n}}}{\frac{1}{n^{3/2}}} = \lim_{n\to 0} \frac{(n-1)n^{3/2}}{n^2\sqrt{n}}$$

$$= \lim_{n\to\infty} \frac{n^{5/2} - n^{3/2}}{n^{5/2}} = \lim_{n\to\infty} \left(1 - \frac{1}{n}\right) = 1.$$

Since the limit is positive and finite, and since $\sum \frac{1}{n^{3/2}}$ converges (p series), then the above sum converges by the limit test.

Note: you could also use the comparison test or the integral test to prove this.

END OF EXAM